Frances Kirwan – my career

1977-81 Undergraduate, Cambridge University
1981-3 DPhil, Oxford (Michael Atiyah)
1983-5 Junior Fellow, Harvard
1985 Visitor, IHES, Paris
1985-6 JRF Magdalen College, Oxford
1986-2017 Tutorial Fellow, Balliol College, Oxford
1988, 1990, 1992 Children born
1994 ICM 1994-2017 Readership
1996-2017 Titular professor
2002 ICM 2003-5 LMS President
2007-11 EWM 2010-16 UKMT Chair
2018 – Savilian professor of Geometry, Oxford

2-quivers and their representations

Joint with Vidit Nanda. No new theorems!

Plan: I Representations of quivers

II 2-categories, $2\mathrm{Vect}_{\mathbb{k}}$, 2-quivers, 2-reps

III Examples

IV Stability conditions

Quiver $Q = \left(Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0 \right)$

$Q_1 = \{\text{edges}\}$

$Q_0 = \{\text{vertices}\}$

s = source

t = target

Quiver $Q = \left(Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0 \right)$

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E.g.

$$\begin{tikzcd}
v_1 \arrow[r, bend left, "e_1"] \arrow[r, "e_2"'] \arrow[dr, "e_5"'] & v_2 \arrow[loop right, "e_3"] \arrow[d, "e_4"] \\
& v_3
\end{tikzcd}$$

$v_1 = s(e_1) = s(e_2) = s(e_3)$

$v_3 = t(e_4) = t(e_5)$

$s = \text{source}$

$t = \text{target}$

Quiver Q $\rightsquigarrow$ Category $\mathrm{Cat}(Q)$

$\mathbin{\rotatebox{60}{$=$}}$

$\left(Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0\right)$

objects are $v \in Q_0$

morphisms are paths in Q

composition is concatenation of paths

Quiver Q $\rightsquigarrow$ Category $\mathcal{C}at(Q)$

$$\left(Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0 \right) = Q$$

objects are $v \in Q_0$

morphisms are paths in Q

composition is concatenation of paths

(linear) rep^n of Q $\leftrightsquigarrow$ functor $\rho : \mathcal{C}at(Q) \to \mathrm{Vect}_{\Bbbk}$

$v \longmapsto W_v$ v space over $\Bbbk$

$e \longmapsto \theta_e : W_{s(e)} \longrightarrow W_{t(e)}$

$\Bbbk$-linear

($\mathrm{Vect}_{\Bbbk}$: category of finite dim^l v spaces over $\Bbbk$)

Coordinatised representation $\rho = (\rho_0, \rho_1)$

$$\rho_0 : Q_0 \longrightarrow \mathbb{N}$$
$$v \longmapsto \mathbb{k}^{\rho_0(v)}$$

If $e \in Q_1$, $\rho_1(e) \in \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{k}}}\left(\mathbb{k}^{\rho_0(s(e))}, \mathbb{k}^{\rho_0(t(e))}\right)$

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Quiver variety (King, ...)

Fix $\rho_0 : Q_0 \to \mathbb{N}$ and consider quotient of

$$\prod_{e \in Q_1} \mathrm{Hom}_{\mathrm{Vect}_{\Bbbk}}\left(\Bbbk^{\rho_0(s(e))}, \Bbbk^{\rho_0(t(e))}\right) \text{ by } \prod_{v \in Q_0} GL(\rho_0(v); \Bbbk)$$

What sort of quotient? Use Mumford's GIT (geometric invariant theory)

What sort of quotient? Use Mumford's GIT (geometric invariant theory)

$$G = \prod_{v \in Q_0} GL(e_0(v); \Bbbk) \quad \text{reductive group}$$

$$\mathbb{A}^N = \prod_{e \in Q_1} \mathrm{Hom}_{\mathrm{Vect}_{\Bbbk}}\left(\Bbbk^{e_0(s(e))}, \Bbbk^{e_0(t(e))}\right) \quad \text{affine space}$$

$$G \circlearrowright \mathbb{A}^N \text{ via } G \to GL(N; \Bbbk)$$

Fix a "stability condition": in this case just a character $\chi : G \to \mathbb{G}_m$ multiplicative group of $G = \prod_{v \in Q_0} GL(\rho_0(v); \mathbb{k})$

which will be given for some $(m_v)_{v \in Q_0} \in \mathbb{Z}^{Q_0}$ by

$$\chi\left((g_v)_{v \in Q_0} \right) = \prod_{v \in Q_0} \det(g_v)^{m_v}$$

GIT quotient:

$$
\begin{tikzcd}
\mathbb{A}^N \arrow[r, dashed] & \operatorname{Proj}\left(\Bbbk[x_1,\ldots,x_N]^{G,\chi}\right) = \boxed{\mathbb{A}^N /\!/_\chi G} \\
\text{open} \ \cup\!| & \cup\!| \ \text{open} \\
(\mathbb{A}^N)^{ss,\chi} \arrow[ur, two heads] & (\mathbb{A}^N)^s / G \\
\text{open} \ \cup\!| & \\
(\mathbb{A}^N)^{s,\chi} \arrow[ur, two heads] &
\end{tikzcd}
$$

semi-invariants

quiver variety

semistable locus

stable locus

geometric quotient

VGIT Variation of GIT describes how $\mathbb{A}^N /\!/_\chi G$ varies as χ varies.

II

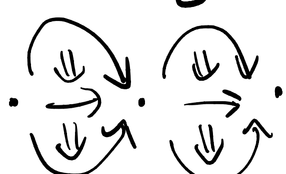
A (strict/weak) 2-category has

objects, $\quad$ 1-morphisms $\quad$ and $\quad$ 2-morphisms

$V \qquad\qquad V \xrightarrow{\theta} W \qquad\qquad$ $\begin{tikzcd} V \arrow[r, bend left, "\theta"{name=U}] \arrow[r, bend right, "\varphi"'{name=D}] & W \arrow[from=U, to=D, phantom, "\Downarrow\varepsilon"] \end{tikzcd}$

(with composition) $\qquad$ (with horizontal & vertical composition)

s.t. the different compositions are associative, unital, and satisfy the interchange law



(in the weak case, only up to natural isomorphisms)

E.g. (1) Cat : <u>objects</u> small categories <u>1-morphisms</u> functors <u>2-morphisms</u> natural transformations

(2) Kapranov - Voevodsky's 2-category $2\mathrm{Vect}_{\mathbb{k}}$:

<u>objects</u> $n \in \mathbb{N}$

<u>1-morphisms</u> from n to m $\begin{pmatrix} V_{11} & V_{12} & \cdots \\ & \cdots & \\ \cdots & & V_{nm} \end{pmatrix}$

$n \times m$ matrices with entries in $\mathrm{Vect}_{\mathbb{k}}$

composition given by matrix mult^n using tensor product and direct sum

<u>2-morphisms</u> $\begin{pmatrix} \ddots & & \\ & V_{ij} & \\ & & \ddots \end{pmatrix} \to \begin{pmatrix} \ddots & & \\ & W_{ij} & \\ & & \ddots \end{pmatrix}$ given by linear maps $V_{ij} \longrightarrow W_{ij}$

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composition given by matrix mult^n using tensor product and direct sum

2-morphisms

$$\begin{pmatrix} & & \\ & V_{ij} & \\ & & \end{pmatrix} \to \begin{pmatrix} & & \\ & W_{ij} & \\ & & \end{pmatrix}$$

given by linear maps $V_{ij} \longrightarrow W_{ij}$

(2′) Coordinatised versions

where 1-morphisms have $V_{ij} = \mathbb{k}^{d_{ij}}$, or

V_{ij} chosen from a fixed list of vector spaces

What's a 2-quiver Q? and a 2-rep^n of Q?

″ (2-) computad (Street 1976) = polygraph

Want $Q \rightsquigarrow 2\mathrm{Cat}(Q)$

2-rep of $Q \rightsquigarrow$ 2-functor $2\mathrm{Cat}(Q) \to 2\mathrm{Vect}_{\mathbb{k}}$

What's a 2-quiver Q? and a 2-rep^n of Q?

$\overset{\shortparallel}{\text{(2-)}}$ computad (Street 1976) = polygraph

Want $Q \rightsquigarrow 2Cat(Q)$

2-rep of $Q \rightsquigarrow$ 2-functor $2Cat(Q) \to 2Vect_{\mathbb{k}}$

<u>Defn:</u> $Q = \Big(\underbrace{(Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0)}_{\text{quiver}}, \underbrace{(\Pi_1 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} \Pi_0)}_{\text{quiver}} \Big)$

s.t. • $\Pi_0 = \text{Paths}(Q_1 \rightrightarrows Q_0)$

• if s, t are extended to $\Pi_0 \rightrightarrows Q_0$ in the obvious way

$s\sigma\varepsilon = s\tau\varepsilon$ and $t\sigma\varepsilon = t\tau\varepsilon$ for all $\varepsilon \in \Pi_1$.

Write Q_2 for Π_1, and $Q = \left(Q_2 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \mathrm{Paths}\left(Q_1 \underset{t}{\overset{s}{\rightrightarrows}} Q_0\right)\right)$

$Q_0 = \{\text{vertices}\}$

$= \{0\text{-edges}\}$ $\qquad \overset{v}{\bullet}$

$Q_1 = \{1\text{-edges}\}$ $\qquad s(e) \bullet \xrightarrow{e} \bullet\, t(e)$

$Q_2 = \{2\text{-edges}\}$

$$\begin{tikzcd}
 & \bullet \arrow[r] & \bullet \arrow[dr] & \\
s\sigma(\varepsilon)=s\tau(\varepsilon)\ \bullet \arrow[ur] \arrow[dr] & & \arrow[d, phantom, "\Downarrow\varepsilon"] & \bullet\ t\sigma(\varepsilon)=t\tau(\varepsilon) \\
 & & \bullet \arrow[ur] &
\end{tikzcd}$$

2Cat(Q) has

objects

vertices

$v \in Q_0$

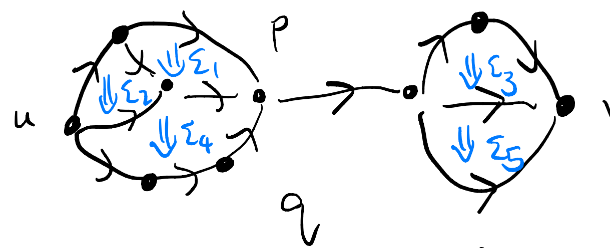
1-morphisms

paths in

$$Q_1 \overset{s}{\underset{t}{\rightrightarrows}} Q_0$$

2-morphisms

equivalence classes of diagrams



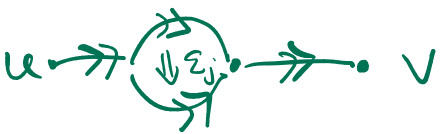
$(\varepsilon_1, \dots, \varepsilon_k)$ 2-edges in Q_2

$p = r_1, r_2 \dots r_k, r_{k+1} = q$

paths in $Q_1 \rightrightarrows Q_0$ from u to v

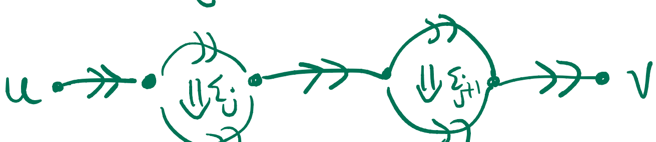
up to "switchability"

At the jth stage



ε_j and ε_{j+1} are switchable

if



2-rep of Q over $\Bbbk$: $\rho = (\rho_0, \rho_1, \rho_2)$

s.t.

- $\rho_0 : Q_0 \to \mathbb{N}$
- $\rho_1 : Q_1 \to \{\text{matrices with entries in } \mathrm{Vect}_{\Bbbk}\}$

  s.t. $\rho_1(e)$ is a $\rho_0(s(e)) \times \rho_0(t(e))$ matrix

[extended to $\mathrm{Paths}(Q_1 \rightrightarrows Q_0)$ via

$$\rho_1\left(v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_m} v_m\right)_{ij} = \bigoplus_{\substack{1 \le k_1 \le \rho_0(v_1) \\ \vdots \\ 1 \le k_{m-1} \le \rho_0(v_{m-1})}} \rho_1(e_1)_{ik_1} \otimes \rho_1(e_2)_{k_1k_2} \otimes \cdots \otimes \rho_1(e_m)_{k_{m-1}j}$$

]

- if $\Sigma \in Q_2$ then $\rho_2(\Sigma)$ is a $\rho_0(v_0) \times \rho_0(v_m)$ matrix

  with $\rho_2(\Sigma)_{ij}$ a linear map from

  $\rho_1(\sigma(\Sigma))_{ij}$ to $\rho_1(\tau(\Sigma))_{ij}$.

2-rep of Q over $\Bbbk$: $\rho = (\rho_0, \rho_1, \rho_2)$

s.t.

- $\rho_0 : Q_0 \to \mathbb{N}$

  from the list $0, \Bbbk, \Bbbk^2, \Bbbk^3, \ldots$

- $\rho_1 : Q_1 \to \{\text{matrices with entries in } \mathrm{Vect}_{\Bbbk}\}$

  s.t. $\rho_1(e)$ is a $\rho_0(s(e)) \times \rho_0(t(e))$ matrix

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from a fixed list $W_0, W_1, W_2 \ldots$

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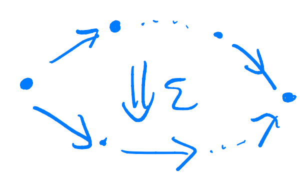

Given a quiver $\left(\tilde{Q}_1 \overset{s}{\underset{t}{\rightrightarrows}} \tilde{Q}_0\right)$,

construct a 2-quiver

$$Q = \left(Q_2 \overset{s_2}{\underset{t_2}{\rightrightarrows}} \mathrm{Paths}\left(Q_1 \overset{s_1}{\underset{t_1}{\rightrightarrows}} Q_0\right)\right)$$

where $Q_0 = \{*\}$

$Q_1 = \tilde{Q}_0 \qquad s_1(v) = * = t_1(v) \ \forall v \in \tilde{Q}_0$

$Q_2 = \tilde{Q}_1 \qquad s_2 = s, \ t_2 = t$

Given a rep^n $(\tilde{\rho}_0, \tilde{\rho}_1)$ of $\left(\tilde{Q}_1 \rightrightarrows \tilde{Q}_0\right)$, construct a 2-rep^n (ρ_0, ρ_1, ρ_2) of Q where

$\rho_0 \equiv 1, \quad \rho_1 = (\tilde{\rho}_0), \quad \rho_2 = (\tilde{\rho}_1)$.

"2-quiver varieties" Fix $W_0, W_1, W_2 \dots$ in $\mathrm{Vect}_{\mathbb{k}}$

Assume $Q = (Q_2 \rightrightarrows \mathrm{Paths}(Q_1 \rightrightarrows Q_0))$ finite

Fix ρ_0 and ρ_1, and consider action of

$$G = \prod_{\ell} GL(W_\ell) \text{ on } \mathbb{A}^N = \{\rho_2\} \text{ affine space}$$

where $GL(W_\ell)$ acts on $\rho_2(\varepsilon)_{ij} : \rho_1(\sigma(\varepsilon))_{ij} \to \rho_1(\tau(\varepsilon))_{ij}$

via its natural action on $\rho_1(e)_{pq}$ when $\rho_1(e)_{pq} = W_\ell$

and the trivial action on $\rho_1(e)_{pq}$ otherwise.

Choose stability parameters $\rightsquigarrow$ GIT quotients

2-quiver varieties

Examples of 2-reps of 2-quivers:

Vector bundles / coherent sheaves

Fix X nonsing. projective variety, and $L_1, \dots, L_k$ ample line bundles over X (cf. Greb-Ross-Toma, 'A master space for moduli spaces of Gieseker-stable sheaves').

Let $Q_0 = \{*\} \sqcup \{E \text{ coherent sheaf over } X\}$

$$Q_1 = \{* \xrightarrow{L_j} E\} \sqcup \{E \xrightarrow{(L_i, L_j)} E\}$$

$$Q_2 = \left\{ \begin{tikzcd} * \arrow[r, "L_j"] \arrow[rr, bend right, "L_k"'] & E \arrow[d, phantom, "\Downarrow"] \arrow[r, "(L_j, L_k)"] & E \\ & {} & \end{tikzcd} \right\}$$

$2\text{-rep}^n: \quad \rho_0(*) = 1, \qquad \rho_0(E) \longmapsto n \gg 1 \ (\text{or } n = \infty)$

$$\rho_1\left(* \xrightarrow{L_j} E\right) = \left(H^0(X, L_j^{\otimes k} \otimes E)\right)_{k=1}^{n}$$

$$\rho_1\left(E \xrightarrow[(L_i, L_j)]{} E\right) = \left(H^0(X, L_i^{-k} \otimes L_j^{m})\right)_{k,m=1}^{n}$$

$$\rho_2\left(\begin{tikzcd} * \arrow[r, "L_i"] \arrow[rr, bend right, "L_j"'] & E \arrow[r, "{(L_i, L_j)}"] \arrow[d, phantom, "\Downarrow"] & E \\ & {} & \end{tikzcd}\right) = \left(\begin{array}{l} \bigoplus_k H^0(X, L_i^{\otimes k} \otimes E) \\ \qquad \otimes H^0(X, L_i^{\otimes(-k)} \otimes L_j^{\otimes m}) \end{array}\right)_{m=1}^{n}$$

$$\Big\downarrow \ \text{natural map}$$

$$\left(H^0(X, E \otimes L_j^{\otimes m})\right)_{m=1}^{n}$$